Algorithms Assignment 1

Your name here & Computing ID

1 Introduction

This assignment is shared between algorithms sections.
Credit: Assit Prof. Brunelle & Assit Prof. Hott

PROBLEM 1 *Asymptotic*

Prove or disprove each of the following conjectures.

1. $2^{n+1} = O(2^n)$.
2. $2^{2n} = O(2^n)$.
3. Given that: $\forall \varepsilon > 0,\ \log(n) = o(n^\varepsilon)$,
   show:
   $\forall \varepsilon, k > 0,\ \log^k(n) = o(n^\varepsilon)$

PROBLEM 2 *Solving Recurrences*

Prove a (as tight as possible) O (big-Oh) asymptotic bound on the following recurrences. You may use any base cases you'd like.

1. $T(n) = 4T(\frac{n}{3}) + n \log n$
2. $T(n) = 3T(\frac{n}{3} - 2) + \frac{n}{2}$
3. $T(n) = 2T(\sqrt{n}) + n$

PROBLEM 3 *Where is Batman when you need him?*

As the newly-hired commissioner of the Gotham City Police Department, James Gordon's first act is to immediately fire all of the dirty cops, stamping out Gotham's widespread police corruption. To do this, Commissioner Gordon must first figure out which officers are honest and which are dirty. There are n officers in the department. The majority ($> n/2$) of the officers are honest, and every officer knows whether or not each other officer is dirty. He will identify the dirty cops by asking the officers, in pairs, to indicate whether the other is dirty. Honest officers will always answer truthfully, dirty cops may answer arbitrarily. Thus the following responses are possible:

Officer A	Officer B	Implication
"B is honest"	"A is honest"	Either both are honest or both are dirty
"B is honest"	"A is dirty"	At least one is dirty
"B is dirty"	"A is honest"	At least one is dirty
"B is dirty"	"A is dirty"	At least one is dirty

1. A group of n officers is uncorrupted if more than half are honest. Suppose we start with an uncorrupted group of n officers. Describe a method that uses only $\lfloor n/2 \rfloor$ pair-wise tests between officers to find a smaller uncorrupted group of at most $\lceil n/2 \rceil$ officers. Prove that your method satisfies each of the three requirements.

2. Using this approach, devise an algorithm that identifies which officers are honest and which are dirty using only $\Theta(n)$ pairwise tests. Prove the correctness of your algorithm, and prove that only $\Theta(n)$ tests are used.

3. Prove that a conspiracy of $\lfloor n/2 \rfloor + 1$ dirty officers (who may share a plan) can foil *any* attempt to find a honest officer. I.e., not only will method above not work, but that there is no way *at all* for Commissioner Gordon to identify even one honest officer if there is not an honest majority.

PROBLEM 4 *Karatsuba Example*

Illustrate the Karatsuba algorithm on 20194102×37591056. Use 2-digit multiplication as your base case.